ANGLES GENERATE MORE FUNDAMENTAL FRAMES THAN LENGTHS

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Abstract

Natural geometry replaces the usual Euclidean model of quaternionic complex numbers by a more general geometric picture. Originally natural geometry is a structure without "straight lines", "unit of lengths" and "Cartesian systems". Trigonometry, founded on this way, leads to two complementary systems of angle measurement. Linear as well as circular angle coordinates can describe quaternionic complex numbers. The two complementary systems of coordinates are connected by "Transformations of Duality". Scalar imaginary unit *i* is the operator of this duality. Make use of linear coordinates must be seen as a non-periodical continuation of trigonometry, classical periodical continuation of trigonometry is based on the circular form of angle measurement. Natural geometry opens a new understanding of Lorentz transformations that can be understood as produced by the non-periodical continuation of trigonometry.

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1. Introduction

This paper shows that elements and algebra of the field **C**, the field of complex numbers, get new aspects if **C** is perceived as a sub field of **H**, the skew field of Hamiltonian quaternions; and if the elements of **H** are perceived as basic figures of *natural geometry*, a conformal, more general form of *three*-dimensional Euclidean geometry [12, 13]. The article is centred on a more general interpretation of the Eulerian numbers $W = \exp(i\varphi)$.

Young R. P. Feynman thought $-1 = \exp(i\pi)$ "the most remarkable formula in math" ([5], page 35). Even 1996 S. de Leo started his article "Quaternions and special relativity" [4] in citing Feynman's "The most remarkable formula in mathematics is $\exp(i\varphi) = \cos\varphi + i.\sin\varphi$. This is our jewel. We may relate the geometry to the algebra by representing complex numbers in a plane $x + iy = r.\exp(i\varphi)$. This is the unification of algebra and geometry" [3]. A more perfect unification of *both* geometry *and* physics with the algebra of numbers can be achieved if complex numbers are perceived as quaternionic numbers that geometrically has to be seen as centred and directed triangles (3-angles, trigons [12, 13]) in the three-dimensional space of our natural perception.

2. The new geometrical understanding of numbers

The traditional picturing of complex numbers in the Gauss/Argand plane and the construction of Hamiltonian quaternions as four-dimensional vectors temps to the prejudice that always the fields of these numbers are Euclidean spaces[1]. 1999 – 2001 I could prove that a conformal form of trigonometry can be constructed with the help of complex numbers but without using Euclidean concepts as "straight line" and "length" [12]. My article [13] shows that complex numbers can be geometrically built in starting with tetraglobes, a con-

[1] Still 1993 R.Anderson and G.C. Joshi wrote, "There has been a long tradition of using quaternions for special relativity…The use of quaternions in special relativity, however, is not entirely straightforward. Since <u>the field of quaternions is a four-dimensional Euclidean space</u>, complex components for the quaternions are required for the 3+1 space-time of special relativity." [1]

form-geometrical basic figure (Definition in Section 2 of [12]) characterized by three angles φ_k.

A complex number Z is a centred and directed tetraglobe ("trigon", Definition in Section 1 of [13]), if non-real described by

$$Z = \frac{\sin\varphi_2}{\sin\varphi_3}.W(\varphi_1) \tag{2.1}$$

with

$$\sum\varphi_k = \pi \tag{2.2}$$

and

$$W(\varphi) := \exp(\boldsymbol{i}\varphi) = \cos\varphi + \boldsymbol{i}.\sin\varphi. \tag{2.3}$$

([12], Theorem 1.1, and [13], Section 1) This geometrical picturing of numbers is possible without using a Cartesian coordinate system. A trigon, constructed by four circles through four points, has a fixed conformal shape, but this figure does not possess a defined magnitude (diameter) or localisation. The sum (2.2) of the three trigon-angles expresses that a trigon is a more general form of a Euclidean triangle. But only in a "special Euclidean location" (Definition 2.3 of [12]) a trigon has the form of a Euclidean triangle[2] (Section 2 of [13]). For instance the imaginary ***i*** appears as an orthogonal and isosceles triangle (with $\varphi_2 = \varphi_3$, $\varphi_1 = \varphi = \pi/2$).

This conformal representation of complex and quaternionic numbers can be seen as a non-Euclidean picture that *only together with additional special conditions* decreases to the traditional Euclidean image of numbers in a Gauss/Argand plane.

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[2] A tetraglobe is the conformal form of a Euclidean triangle (with circum circle). To underline this relationship as well as the difference, a tetraglobe is also called "3-angle". A trigon is the conformal form of a centred and directed Euclidean triangle. A triangle is centred and directed if one of the 2 x 3 permutations of its 3 angles is fixed and used. – Sometimes I colloquially use the term "triangle" in the sense of "trigon".

The real number $\sqrt{Z.Z*}$, associated to every complex number *Z*, constructed by *Z* and the conjugate *Z**, cannot generally seen as the magnitude (quantity) of a pointer (vector) *Z* if a unit of length in a Cartesian system does not exist. The more general conformal picture is associated with

$$\sqrt{Z.Z*} = \frac{\sin\varphi_2}{\sin\varphi_3}, \qquad (2.4)$$

a part of the conformal sine law (Details in Section 9 of [12]). Generally $\sqrt{Z.Z*}$ is not the length of the hypotenuse of a right-angled triangle (Cf. Section 10 of [12]). In this sense Pythagoras' theorem is not generally valid in natural geometry; a number field does not generally be a Euclidean space.

I call the conformal form of elementary geometry with its conformal form of trigonometry, in details developed in [11, 12], "*Natural geometry*" because in this geometry only the measurement of angles is defined. A unit for measurement of lengths does generally not exist. This is a natural precondition because science possesses a natural unit of angles. Mathematics and physics do not possess a natural unit of lengths.

3. On the naturalness of complex numbers

The foundation of natural geometry and its trigonometry, without using the concept of "length", puts a new complexion on the base of complex numbers:
In constructing complex numbers as Cartesian pairs mathematics *implicitly* uses a Cartesian system *and the units of lengths on its axes.* If the construction of complex numbers is started with the help of trigons it is evident that the concept "length" is not a part of a pure mathematical concept of these numbers. Together with special conditions it is possible to give a number *Z* a locus (point) in a plane with the help of a Cartesian system, but originally a trigon *Z* does not possess a fixed location in a Euclidean plane (the Gauss/Argand plane). It is possible to construct also real numbers ("degenerated trigons", cf. Remark 1, Section 2 of [12]) starting with genuine trigons (non-real numbers,

non-degenerated trigons). For instance, if the trigon Z is the imaginary unit $\boldsymbol{i}$, the real numbers –1 and +1 can be constructed with the help of products by

$$-1 := \boldsymbol{i}^2 \quad \text{and} \quad +1 := \boldsymbol{i}.\boldsymbol{i}^* . \tag{3.1}$$

On this way all real numbers can be produced as products of (for instance orthogonal) trigons. But natural geometry does not possess straight lines and lengths on these lines. Natural geometry cannot illustrate these special complex numbers, namely the real numbers on a Euclidean number line. If complex numbers are constructed on this way non-real numbers seems more original than real numbers. Non-real numbers are genuine trigons; real numbers are degenerated trigons.

Conceptually we have to differ between the algebraic field $\mathbf{C}_{\boldsymbol{i}}$, which possesses elements Z seen (only) as trigons, and the traditionally seen algebraic field $\mathbf{C}_i$. Because the elements Z of the classical field $\mathbf{C}_i$ are especially seen as elements *in* a plane described by a Cartesian system, these numbers can be pictured by pointers (vectors) and points of these (Gauss) plane.

For the following I emphasize the difference of the "conformal" imaginary trigon $\boldsymbol{i} \in \mathbf{C}_{\boldsymbol{i}}$, with $\boldsymbol{i}^2 = -1$, and the traditional "Euclidean" imaginary unit $i \in \mathbf{C}_i$, with $i^2 = -1$,

$$i \neq \boldsymbol{i}. \tag{3.2}$$

It is important and necessary to see the difference of $\mathbf{C}_{\boldsymbol{i}}$ and $\mathbf{C}_i$ and the difference of $\boldsymbol{i}$ and i if we want to realize that the theorem "the fields of numbers are Euclidean spaces" is a prejudice. But on the other side both structures $\mathbf{C}_{\boldsymbol{i}}$ and $\mathbf{C}_i$ are identical on the abstract level for both structures are isomorphic algebraic fields.

Both structures are isomorphic, how can we geometrically perceive the correspondence between both structures? Generally the numbers of $\mathbf{C}_{\boldsymbol{i}}$ are elements on the surface of a conformal sphere; the elements of $\mathbf{C}_i$ are located in a Euclidean plane (Gauss plane). If all trigons of $\mathbf{C}_{\boldsymbol{i}}$ are especially in a Euclidean location (Cf. Definition 2.3 of [12]) their conformal sphere looks also like a plane. We may identify this ‘plane’ of $\mathbf{C}_i$ and the Gauss plane of $\mathbf{C}_i$. Now the Cartesian system of the Gauss plane can be used to link analogous elements of both algebraic structures: If three of the four points of a tetraglobe Z are identified with the points 0, 1 and ∞ of the Cartesian system the fourth point of the Gauss plane is associated with Z. For example the right-angled and isosceles tetraglobe $Z = \boldsymbol{i} = W(\pi/2)$ is in correspondence with the point (0, 1) = i of the Gauss plane.

4. Complex numbers perceived as quaternions

The trigon picture also proves its worth in seeing quaternions as a direct generalization of common complex numbers.

Hamilton had much trouble to construct the elements of his quaternionic skew field **H** and to define their multiplication because he could not find a point (or vector) picture of these elements in the three-dimensional space of our visual perception. Such an image does not exist. Often quaternions are seen as vectors of a *four*-dimensional space. So we have a curious gab between the common complex numbers in two dimensions and the Hamiltonian quaternionic numbers in four dimensions.

But we get a picture of quaternions in the *three*-dimensional space of our perception if these numbers are perceived as trigons [10, 13]. Common complex numbers can - in special Euclidean location - be seen as centred and directed triangles in *one* plane. Two *genuine* quaternions $Z_1, Z_2 \in \mathbf{H}$ with a non-commutative multiplication, $Z_1.Z_2 \neq Z_2.Z_1$, are complex numbers in *different*, non-parallel (Gauss) planes of the natural three-dimensional space. The multiplication of two quaternionic numbers can be visualized by the composition of two triangles. (Article [10] has proved details of this picture in Euclidean location, for details in general location see [13]) So we come from the common complex numbers to the complete quaternionic number system if not only numbers in one plane $\mathbf{C}_i$ of the natural space but in all planes of the natural three-dimensional space are accepted as numbers $Z \in \mathbf{H}$.

I see the imaginary unit $\boldsymbol{i}$ of every plane $\mathbf{C}_{\boldsymbol{i}}$, the orthogonal and isosceles trigon (triangle) in this plane, as the geometrical figure that characterized the position (direction) of this plane $\mathbf{C}_{\boldsymbol{i}}$ in the natural space. Often we may also perceive this unit $\boldsymbol{i}$ as a "quaternionic vector" orthogonal to $\mathbf{C}_{\boldsymbol{i}}$. But the difference of a common vector and a quaternionic vector should be seen: A common vector possesses a fixed length. Originally, on the level of natural geometry a quaternionic vector has a non-defined length.

If quaternions are (also) written as a sum of a "real part" τ and an "imaginary part"[3] x,

$$Z = \tau + x, \tag{4.1}$$

we get a good algebraic expression of the fact that quaternionic numbers are a very direct generalisation of common complex numbers. τ is a real number, x an orthogonal trigon. In this form we have still a description of complex numbers on the level of natural geometry, the imaginary part x has a well-defined shape but it has a 'non-fixed' length. We can go back to the usual length-metrical level in seeing x as a common vector, for instance by the specification

$$x = \boldsymbol{i}.x\wedge, \tag{4.2}$$

with $x\wedge$ as the length (amount, quantity) of the vector x.

It is $Z = \tau + x \in \mathbf{C} = \mathbf{C}_{\boldsymbol{i}}$ if $\boldsymbol{i}$ is a vector of one dimension. Generally in $\mathbf{H}$ the imaginary x and the unit $\boldsymbol{i}$ has three dimensions.

5. Algebraic elements of numbers and triangles

In the following every quaternion $Z = \tau + x \in \mathbf{H}$ shall possess the *same* direction $\boldsymbol{i}$. We only stay in a subset $\mathbf{C} \subset \mathbf{H}$. Because all numbers have the same $\boldsymbol{i}$ the multiplication of elements $Z = \tau + x = \tau + \boldsymbol{i}.x\wedge \in \mathbf{C} = \mathbf{C}_{\boldsymbol{i}}$ is always com-

[3] Students who do not know the usual concept of quaternions may read for example the website of E. W. Weisstein, http://mathworld.wolfram.com/Quaternion.html. His formulae (25), (26) correspond to our view of a quaternionic number as a pair (τ, x) of a real τ and a quaternionic vector x.

mutative. **C** is a commutative sub field of **H**, a field of the classical complex numbers.

$$Z^* := \tau - x \tag{5.1}$$

is the conjugate of *Z*. In the following the three numbers ρ, *W* and *v* are specially used:

$$\rho := \sqrt{Z.Z^*} = \sqrt{\tau^2 - x^2} \tag{5.2}$$

$$W := \frac{Z}{\rho}, \qquad W.W^* = 1 \tag{5.3}$$

$$v := \frac{x}{\tau} \tag{5.4}$$

The connection of *W* and *v* is described by

$$W = \frac{1+v}{\sqrt{1-v^2}}, \qquad v = \frac{W - W^*}{W + W^*}. \tag{5.5}$$

Remark 2.1
The traditional description of *Z.Z** by $Z.Z^* = \tau^2 + x^{\wedge 2}$ is a description that is *non-free* of the *length-metrical coordinate* $x^\wedge$ of $x = \boldsymbol{i}.x^\wedge$.

The particular number $Z = W(\varphi)$ is characterized by $\varphi_2 = \varphi_3$, $\varphi_1 = \varphi$. (Cf. (2.1)) In special Euclidean location (Definition 2.3 of [12]) the number *W* appears as an isosceles triangle.

The particular number $Z = v(\varphi)$ of (5.4) is a right-angled triangle:

$$\varphi_1 = \frac{\pi}{2}, \qquad \varphi_2 = \varphi, \qquad \varphi_3 = \frac{\pi}{2} - \varphi, \tag{5.6}$$

so that with (2.1)

$$v = \frac{\sin\varphi}{\sin(\pi/2 - \varphi)}.W(\pi/2) \;=\; \boldsymbol{i}.\tan\varphi\,, \tag{5.7}$$

([12]. Section 5). This corresponds to (5.5)

$$v = \frac{W - W^*}{W + W^*} \;=\; \boldsymbol{i}.\tan\varphi \tag{5.8}$$

with $W(\varphi) := \exp(\boldsymbol{i}\varphi)$. As usual we get

$$Z = \tau + x = \rho.W(\varphi) = \rho.\cos\varphi + \boldsymbol{i}.\rho.\sin\varphi\,, \tag{5.9}$$

$$\tau = \rho.\cos\varphi\,, \quad x = \boldsymbol{i}.\rho.\sin\varphi = \boldsymbol{i}.x\wedge, \quad x\wedge = \rho.\sin\varphi\,. \tag{5.10}$$

But we understand Feynman’s “unification of algebra and geometry” in a modified sense: On the level of natural geometry numbers and (directed and centred) triangles seem two aspects of the same entity. For example we may perceive the formulae (5.5) as two equations that describe the connection of the two numbers *W* and *v*. Or we may see (5.5) as the connection of an isosceles and an orthogonal triangle. In my eyes the formulae (5.5) are an excellent example of Feynman’s unification because the angle φ, which implicitly donates the relationship of the figures *W* and *v*, does not explicitly appear in these equations. These formulae describe a geometrical situation but *without measurements of angles and lengths*; *without using a Cartesian system*.

Seen from the level of natural geometry the “unification of algebra and geometry” is the outstanding possibility to unify the theory of quaternionic complex numbers with the theory of (conformal) triangles. If this unification (of “Zahlenlehre und Trigonometrie”) shall be carved out we should not rashly identify the algebraic term $Z.Z^* = (\tau + x)(\tau - x) = \tau^2 - x^2$ and Pythagoras’ geometrical theorem. On the genuine conformal level of natural geometry Pythagoras’ theorem is not valid in its classical form (Cf. [12], Section 10).

6. The E-function

With (5.5) the elementary but fundamental function on **H**

$$Z \;\rightarrow\; Z' = W.Z, \quad W.W^* = 1, \tag{6.1}$$

can be split in its real and imaginary components without using any coordinate system. If

$$Z = \tau + x, \qquad Z' = \tau' + x' \tag{6.2}$$

one gets the

real part
$$\tau' = \frac{\tau + v.x}{\sqrt{1-v^2}} \tag{6.3}$$

and the imaginary part
$$x' = \frac{x + v\tau}{\sqrt{1-v^2}} \tag{6.4}$$

of (6.1).

An invariant of this transformation is

$$Z.Z^* = \tau^2 - x^2 = \tau'^2 - x'^2 = Z'.Z'^*. \tag{6.5}$$

Remarks 6.1

1. This component description of $Z \rightarrow Z'$ has the *shape* of ***Lorentz transformations*** if light velocity *c* is standardized by $c^2 = 1$: With *x* as locus and τ as time, *v* describes the velocity.

2. The traditional description of $Z.Z^*$ by $Z.Z^* = \tau^2 + x^{\wedge 2}$ is a description that is *non-free* of the length-metrical *coordinate* $x^\wedge$. The formulae (6.3) and (6.4) are a *component* description of (6.1) but this component description is free of *coordinates,* which describe *x.*

The transformations (6.3) and (6.4) have the mathematical pattern of Lorentz transformations. This pattern emphasized that an "entirely straightforward use of quaternions in special relativity" (missed by Anderson and Joshi, cf. footnote 1) is possible if we see, accept and use the natural geometric level of numbers. But (6.3) and (6.4) has only the *form* of Lorentz transformations without the physically essential *measurement* of quantities as τ, x and v.

7. The linear and circular figures of Eulerian numbers

In the following I use *two systems* of measuring angles, *two* angle coordinate systems.
The two coordinate systems $^\wedge\varphi$, $\varphi^\wedge$ are connected by

$$^\wedge\varphi = i.\varphi^\wedge , \qquad \varphi^\wedge = i^*.^\wedge\varphi . \tag{7.1}$$

First time here I employ the element i with $i^2 = -1$. i is not an element of $\mathbf{C} = \mathbf{C}_{\boldsymbol{i}}$, $i \neq \boldsymbol{i}$. In the following the imaginary i is used to describe the *connection* between the *coordinates* $^\wedge\varphi$ and $\varphi^\wedge$ of the numbers $Z \in \mathbf{C}$.[4]

In using $\varphi^\wedge$ we get the *traditional figure* of the Eulerian numbers W with $W.W^* = 1$, namely

$$W(\varphi) := W^\wedge = \cos\varphi^\wedge + \boldsymbol{i}.\sin\varphi^\wedge = \exp(\boldsymbol{i}\varphi^\wedge) . \tag{7.2}$$

In using $^\wedge\varphi$ we get the *complementary figure* of these Eulerian numbers

$$W(\varphi) := {}^\wedge W = \cosh{}^\wedge\varphi + \mathbf{1}.\sinh{}^\wedge\varphi = \exp(\mathbf{1}{}^\wedge\varphi) , \tag{7.3}$$

if

$$\mathbf{1} := \boldsymbol{i}.i^* , \tag{7.4}$$

[4] Definition 5.3 in article [13] gives a geometrical definition of the scalar i that does not depend on the length unit of a Cartesian system.

and

$$\sin\varphi\text{^} = i^* \sinh\text{^}\varphi\,, \tag{7.5}$$

$$\cos\varphi\text{^} = \cosh\text{^}\varphi\ . \tag{7.6}$$

With (7.4) it follows $\mathbf{1}^2 = +1$. The difference of 1 and **1** has to be seen. It is $\mathbf{1}^2 = +1$ but it is $1 \neq \mathbf{1}$. **1** may be called "real" but the *vector* **1**, orthogonal to the triangle numbers *W* (and every 3-angle-number *Z* of **C)**, does not be a "real number" in the traditional sense. It is an element of the imaginary, not of the real part of a quaternion.

In using (7.1) it follows (7.5), (7.6) with the help of the known connections of circle and hyperbola functions proved by complex function theory (Cf. Section 8).

With the preconditions (7.4), (7.5) and (7.6) the natural geometry of trigons (triangles) as well as the structure **H** of numbers possess *two* figures *W*^ , ^*W* of the Eulerian *W*.

If the description of **H**-elements with the help of two complementary angle coordinates φ^ and ^φ is accepted, we also get *two* "length-metrical" coordinates *x*^ and ^*x* with

$$\text{^}x := i.x\text{^}, \qquad x\text{^} = i^*.\text{^}x\,, \tag{7.7}$$

so that

$$Z = \tau + x = \tau + \boldsymbol{i}.x\text{^} = \tau + \mathbf{1}.\text{^}x\ . \tag{7.8}$$

I call a coordinate system (^φ, ^*x)* a *linear,* a system (φ^, *x*^*)* a *circular* or *traditional* coordinate system. I call the transformations (7.1), (7.4) - (7.7), which describe the connection between these *complementary coordinates* the *transformations of duality*.

Remark 7.1
The transformations of duality, collected in (8.6), are not parts of the algebra in **H**. These equations only establish the *connection* between the two complementary coordinate descriptions of the elements of **H**. The operator *i*, *which produces the transformation of duality,* is *not appearing explicitly somewhere* if only *either* the circular *or* the linear coordinate system is used.

8. On the measurement and the functions of angles

Trigonometry of natural geometry defines the measurement of angles with the help of tetraglobes [12], traditional trigonometry with the help of Euclidean triangles. Angles of conformal as well as angles of Euclidean triangles are originally restricted to

$$0 < \varphi < \pi . \tag{8.1}$$

Trigonometry of natural geometry defines the 2 x 3 trigonometric functions with the help of the 2 x 3 permutations v_k of the characteristic number (metrical number) v of a right-angled trigon (Definition 5.2 of [12]): Here as well as in the usual traditional trigonometry the domains of the three angle functions $\tan\varphi$, $\cos\varphi$ and $\sin\varphi$ are restricted to $0 < \varphi < \pi$.

Also the Eulerian figure (2.3)

$$W(\varphi) = \exp(\boldsymbol{i}\varphi) = \cos\varphi + \boldsymbol{i}.\sin\varphi \tag{8.2}$$

is restricted to $0 < \varphi < \pi$ if W is seen as a triangle. (Cf. Theorem 1.1 of [12])

I call (8.1) the "trigonometric restriction of angles and angle functions". For instance this restriction induces $1 \geq \sin\varphi \geq 0$. Mathematics and physics have not respected this trigonometric restriction. Any real number, $\varphi \in \mathbf{R}$, is accepted as argument of (8.2) and as argument of any trigonometric function.

I call this more general concept of angles and angle measurement the “classical non-trigonometric (trans-trigonometric) angle concept”.

Often an angle and the measuring number of this angle are named with the same sign. Now only I will use ‘φ’ as name for an angle to avoid this ambiguity in the following sections. The signs φ^ , ^φ describe the measuring numbers of φ.

In Section 7 two complementary figures of (8.2) were developed, namely

$$W(\varphi)= W\hat{} := \cos\varphi\hat{} + \boldsymbol{i}.\sin\varphi\hat{} = \exp(\boldsymbol{i}\varphi\hat{})\ , \qquad (8.3)$$

$$W(\varphi)= \hat{}W := \cosh\hat{}\varphi + \mathbf{1}.\sinh\hat{}\varphi = \exp(\mathbf{1}\hat{}\varphi)\ . \qquad (8.4)$$

That the classical trans-trigonometric angle concept again - the second time - is generalized by

$$\varphi\hat{},\ \hat{}\varphi \ \in\ \mathbf{C}_i \qquad (8.5)$$

is the precondition of this description of *W* with the help of *two* complementary angle coordinates. φ^, ^φ are (real or imaginary) *measuring numbers* for describing elements of $\mathbf{C} \subset \mathbf{H}$. The scalar *i* is the imaginary unit of $\mathbf{C}_i$, it is $\mathbf{C}_i \neq \mathbf{C}$. Only if the angle coordinates are elements of the *complex* set $\mathbf{C}_i$ the equations of transformation (7.5), (7.6) can deduced with the help of series in this complex domain.

The classical trans-trigonometric angle concept is a natural continuation of the elementary, trigonometric concept if $\varphi\hat{} \in \mathbf{R}$ and if *W* is described by *W*^. (**R** is the set of real numbers)

The non-classical trans-trigonometric angle concept is a natural continuation of the elementary, trigonometric concept if $\hat{}\varphi \in \mathbf{R}$ with *W* described by ^*W*.

Mathematics and physics do not possess a natural unit for the measuring of lengths. (For example “meter” or “foot” are artificial units) But science possesses a natural measurement unit (scale unit) of angles. This “full turn” can be described with the number 2π (or $h/2\pi = 1$) as well as with the number $2\pi i$ (or $h/2\pi = i$). Trigonometry defines the measurement of angles in and with

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(8.6)

Transformations of Duality

Coordinate Systems

↓ **circular**	↓ **linear**
$\varphi\wedge = i^*.\wedge\varphi$	$\wedge\varphi = i.\varphi\wedge$
$x\wedge = i^*.\wedge x$	$\wedge x = i.x\wedge$,

Trigonometric Functions

$\tan\varphi\wedge = i^* \tanh\wedge\varphi$	$\tanh\wedge\varphi = i \tan\varphi\wedge$
$\sin\varphi\wedge = i^* \sinh\wedge\varphi$	$\sinh\wedge\varphi = i \sin\varphi\wedge$
$\cos\varphi\wedge = \cosh\wedge\varphi$	$\cosh\wedge\varphi = \cos\varphi\wedge$

Imaginary Vector Units

$\boldsymbol{i} = \mathbf{1}.i \qquad \mathbf{1} = \boldsymbol{i}.i^*$

Units of Natural Angle Scale

$h\wedge = 2\pi$ ← **periodic** → $\wedge h = 2\pi.i$

$h\wedge = 2\pi.i^*$ ← **non-periodic** → $\wedge h = 2\pi$

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the help of triangles. But trigonometry does not define *units* of angle measurement. Trigonometry permits both $\varphi\wedge \in \mathbf{R}$ and $\wedge\varphi \in \mathbf{R}$. (Cf. [12], Section 11)

I call the description of W with $\varphi\wedge \in \mathbf{R}$ the circular and periodic description of W, the description of W with $\wedge\varphi \in \mathbf{R}$ the linear and non-periodic description of W. „Circle“ functions, used to describe W, are periodic functions if $\varphi\wedge \in \mathbf{R}$. „Hyperbolic“ functions, used to describe W with the help of $W\wedge$, are non-periodic functions if $\wedge\varphi \in \mathbf{R}$.

At first natural geometry only introduces and uses two *angle* coordinate systems. Generally Cartesian coordinates with their units of length are not introduced. Only in special situations also Cartesian coordinates (7.7)

$$\wedge x = i.x\wedge, \qquad x\wedge = i^*.\wedge x\ ,$$

may be used. Angles generate more fundamental frames of coordinate than lengths.

Does the description of W (and all numbers $Z \in \mathbf{C}$) with two complementary coordinates only be an isomorphic (and perhaps trivial) doubling of the usual, traditional description with the help of only circular coordinates? The complementary description of W is an essential doubling because it makes manifest that the elementary trigonometric definition of angle measurement allows a “periodic continuation” as well as a “non-periodic continuation”. The continuation is non-periodic if $\wedge\varphi = i.\varphi\wedge$ is real; the continuation is periodic if $\varphi\wedge = i^*.\wedge\varphi$ is real.

9. **E-function in complementary coordinates**

Traditionally everybody thinks the function $Z \rightarrow Z' = W.Z,\quad W.W^* = 1$, as a rotation in the Gauss plane. We reflect the preconditions of this thinking.

This article started with the additive representation

$$Z = \tau + x \tag{9.1}$$

of a number/triangle *Z*. Somebody may think that this is an incomplete representation without the circular coordinates *x^*, which opens an explicit appearing of the imaginary unit ***i*** by $x = \boldsymbol{i}.x^\wedge$. But in some situations also this representation of *Z* by a pair (τ, x) is inconvenient, unsuitable or secondary. Natural geometry describes a triangle $Z = (\varphi_1, \varphi_2, \varphi_3)$ by

$$Z = \sin^{-1}(\varphi_3{}^\wedge).\sin(\varphi_2{}^\wedge).W^\wedge(\varphi_1{}^\wedge) \quad \text{or} \quad Z = \sinh^{-1}({}^\wedge\varphi_3).\sinh({}^\wedge\varphi_2).{}^\wedge W({}^\wedge\varphi_1) \qquad (9.2)$$

(Cf. equation (2.1), Section 2). On this conformal level of natural geometry $\sqrt{Z.Z^*}$ cannot generally seen as a length-metrical quantity (scalar) ρ. Generally a unit of length ρ_0 does not exist. Natural geometry understands the *conformal* invariant *Z.Z** as a manifestation of the sine-theorem of Section 9, [12], expressing that the shape of a triangle $Z= (\varphi_k)$ is determined by the ratio of $\sin(\varphi_3{}^\wedge)$ and $\sin(\varphi_2{}^\wedge)$ as well as by the ratio of $\sinh({}^\wedge\varphi_3)$ and $\sinh({}^\wedge\varphi_2)$.

*W.W**=1 does not generally express that a pointer *W* has the length 1. A conformal triangle *W* with *W.W**=1 is an isosceles triangle. The size (magnitude) of this triangle is not defined on the conformal level.

I will specify:
The E-function $Z \rightarrow Z' = W.Z, \quad W.W^* = 1$, can be perceived as a rotation

1. If a Gauss plane exists with a complete Euclidean structure and its length unit ρ_0 ,
2. If the elements of **C** are represented by circular coordinates $\varphi^\wedge, x^\wedge, v^\wedge$,
3. If the measuring numbers $\varphi^\wedge$ of the trigonometric angles φ are chosen real,
4. If the angle concept is extrapolated to a trans-trigonometric one with $\varphi^\wedge \in \mathbf{R}$.

In using the coordinate description of *x* and *v*

$$x = \boldsymbol{i}.x^\wedge, \qquad v = \boldsymbol{i}.\tan\varphi^\wedge \qquad (9.4)$$

we get, starting with (6.3) and (6.4), the usual coordinate description of rotations

$$\tau' = \tau.\cos\varphi\hat{} \; - x\hat{}.\sin\varphi\hat{}$$

$$x\hat{}' = \tau.\sin\varphi\hat{} + x\hat{}.\cos\varphi\hat{}. \qquad (9.5)$$

In the following I will presuppose:

1. A special physical situation may exist that allows the using of a time unit ρ_0 ,
2. In this situation a physical transformation or movement can be described by complex numbers represented by linear coordinates $\hat{}\varphi$, $\hat{}x$, $\hat{}v$,
3. The angles φ of triangles Z will be measured with real $\hat{}\varphi$,
4. The angle concept is extrapolated to a trans-trigonometric one with $\hat{}\varphi \in \mathbf{R}$.

With these preconditions, $v = \mathbf{1}.\hat{}v$ and $x = \mathbf{1}.\hat{}x$, scalar and vector parts (6.3), (6.4) of the E-function get the form

$$\tau' = (\tau + \hat{}v.\hat{}x).\gamma$$

$$\hat{}x' = (\hat{}x + \hat{}v.\tau).\gamma, \qquad \gamma = \frac{1}{\sqrt{1-\hat{}v^2}}. \qquad (9.6)$$

These *are* the **Lorentz transformations** in their most usual form.

In using $v = \mathbf{1}.\tanh\hat{}\varphi$ they can also be written

$$\tau' = \tau.\cosh\hat{}\varphi + \hat{}x.\sinh\hat{}\varphi$$

$$\hat{}x' = \tau.\sinh\hat{}\varphi + \hat{}x.\cosh\hat{}\varphi \;. \qquad (9.7)$$

I want to describe the relationship of the E-function to a rotation and a Lorentz transformation:
If the E-function is written in using the additive component representation of numbers, this function has the *form* (6.3), (6.4) of a Lorentz transformation. The E-function is neither a rotation nor a Lorentz transformation for these geometrical respective physical transformations are defined in spaces that possess a length-metrical unit (scale or scalar unit) ρ_0. But the E-function is a pure mathematical one. Functions of the pure mathematical **H** do not possess lengths.

In other connections, seeing numbers as physical quanta [14], I have explained that an E-function can also interpreted physically (geometrically) *without* using a unit of lengths.

An E-function may also perceived as an entity that "in its mathematical nucleus" is both a rotation and a Lorentz transformation because it can be completed in both directions by using the right scales and the right coordinates: Rotations are a periodical, Lorentz transformations a non-periodical *metrical continuation* of E-functions.

My results are collected in three theorems:

Theorem 9.1

The E-function $Z \rightarrow Z' = W.Z$, $W.W^* = 1$, on **H** can be seen as a Lorentz transformation if the trigonometry of the numbers $Z, W, x, v \in \mathbf{H}$ is non-periodically continued by using angle- and length-metrical coordinates $\hat{}\varphi$, $\hat{}x$, $\hat{}v$ so that $x = \mathbf{1}\hat{}x$, $v = \mathbf{1}\hat{}v$, $v = \mathbf{1}.\tanh\hat{}\varphi$.
A complementary interpretation of this E-function is reached if the trigonometry of these numbers $Z, W, x, v \in \mathbf{H}$ is periodically continued by using angle- and length-metrical coordinates $\varphi\hat{}$, $x\hat{}$, $v\hat{}$ so that $x = \boldsymbol{i}.x\hat{}$, $v = \boldsymbol{i}.v\hat{}$, $v = \boldsymbol{i}.\tan\varphi\hat{}$.

I read the abbreviation "E-function" both as <u>E</u>instein- and <u>E</u>uler-Function.

Theorem 9.2

On the conformal level of natural geometry the Euclidean trigonometry of triangles and numbers can be formulated without settling a unit of length. It depends on the description of the natural angle unit either by $h^\wedge = 2\pi$ or by $^\wedge h = 2\pi$ whether the trigonometric functions W and v appear in the forms $W(\varphi) = \exp(\boldsymbol{i}\varphi^\wedge)$, $v(\varphi) = \boldsymbol{i}.\tan\varphi^\wedge$ or in the forms $W(\varphi) = \exp(\mathbf{1}^\wedge\varphi)$, $v(\varphi) = \mathbf{1}.\tanh^\wedge\varphi$. Choice of $h^\wedge = i{*}^\wedge h = 2\pi$ leads to a periodical, choice of $^\wedge h = ih^\wedge = 2\pi$ to a non-periodical continuation of trigonometric angle measurement.
The choice of angle measurement, either by $h^\wedge$ or $^\wedge h$, induces the measuring of the imaginary part x of a triangle (trigon, number) $Z = \tau + x$. The angle measure $h^\wedge$ leads to a real coordinate $x^\wedge$, the natural measure $^\wedge h$ to an imaginary coordinate $^\wedge x$ of the number part x. Angles φ and their natural unit generate more fundamental frames than the “space part” x.

Theorem 9.3

Formulae (5.5)

$$W = \frac{1+v}{\sqrt{1-v^2}}, \qquad v = \frac{W - W^*}{W + W^*} \tag{9.8}$$

describe the connection between the basic “complex” trigonometric functions $W(\varphi)$ and $v(\varphi)$.
This form of description does not depend on the choice of coordinates $^\wedge\varphi$ or $\varphi^\wedge$. These formulae *implicitly* express the correlations of the circular trigonometric functions (“circle functions”) $\{\sin\varphi^\wedge, \cos\varphi^\wedge, \tan\varphi^\wedge\}$ as well as the correlations of the linear trigonometric functions (“hyperbolic functions”) $\{\sinh^\wedge\varphi, \cosh^\wedge\varphi, \tan^\wedge\varphi\}$.[5]

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[5] For example the first equation defines $\tan\varphi^\wedge$ with the help of $\sin\varphi^\wedge$ and $\cos\varphi^\wedge$; the second equation defines $\sin\varphi^\wedge$ and $\cos\varphi^\wedge$ by $\tan\varphi^\wedge$. Not only $v = v_1$ but all 2 x 3 permutations v_k of v (in the sense of (5.3) of [12]) are describable as functions of the angle-parameter W.

The Transformations of Duality (8.6) describe the correlations of the circular and linear parameters *on the metrical level of coordinates*.

Remarks 9.1

1. Physical textbooks mention and use the "hyperbolic" form of Lorentz transformations (9.7) (for example Dirac [2], § 67 (19), Green [6], p. 104). My new aspect is: Lorentz transformations are a special moulding of a function $Z \to Z' = W.Z$ on $\mathbf{C} \subset \mathbf{H}$, if $Z, W \in \mathbf{C}$ *are described with linear and non-periodical coordinates*. Pure mathematics is roomy enough to allow this form of coordinate description together with a complementary form, which uses circular and periodical coordinates.

2. With the help of the operator *i* the connection of linear and circular coordinates can be formulated. But *i* is not appearing *explicitly* in the linear, non-periodical system (or in the circular, periodical system). If relativity is only written in this linear system nobody sees the "imaginary background" of the coordinates ^*x* in (9.6) and (9.7). Only together with the operator *i* and its coordinate transformations we possess the *coordination* of a periodical and a non-periodical aspect of mathematical numbers *Z*, geometrical triangles *Z* and physical quanta *Z*. Here the *duality* of physical movements has its radix.

3. My article may be seen as a deduction of physical Lorentz transformations out of a more general context: "Lorentz transformation" is a special interpretation of the function $Z \to Z' = W.Z$ on **H**. This function $Z \to Z'$ is a special function on **H**. "Lorentz transformation" describes a coordinate transformation but also a very special physical movement, namely inertia movements. If the special function $Z \to Z' = W.Z$ on **H** can be seen as a special but very basic physical movement also *more general* functions $Z \to Z' = f(Z)$ on **H** may describe physical movements because elements $Z \in \mathbf{H}$ can be perceived as physical quanta [14] in the natural three-dimensional space of our experiences:

My hypothesis is that a special set of functions $Z \rightarrow Z' = f(Z)$ describes movements of these quanta.

My results provoke a reflection upon the geometrical *and* physical meaning of the angle units {*h*^, ^*h*} and the imaginary ('space') units {***i***, **1**}. *This* primarily mathematical article will *not* deepen my physical interpretation by action quantum and light velocity [10, 14].

10. Summary

The traditional description and representation of complex numbers, the usual geometrical interpretation and picturing of **C** and **H** have several handicaps. They do not sufficiently distinguish between

- The vector ***i*** and the scalar *i* ,
- The elements of $\mathbf{C}_{\boldsymbol{i}}$ and the elements of $\mathbf{C}_i$,
- The representation of numbers by two components or their description by coordinates,
- The linear and the circular coordinate representation,
- The Euclidean point and the conformal triangle picture,
- The settlement of numbers in Euclidean planes and in conformal spheres,
- The description of the complex figure *W* either with the help of *W*^ or ^*W*,
- The interpretation of $\sqrt{Z.Z^*}$ as a conformal invariant or as a length-metrical one,
- The interpretation of $\sqrt{Z.Z^*}$ with or without the help of Pythagoras' theorem,
- The physical (geometrical) interpretation of the function $Z \rightarrow Z' = W.Z$, $W.W^* = 1$, in using a linear or a circular coordinate system.
- The physical (geometrical) interpretation of the function $Z \rightarrow Z' = W.Z$, $W.W^* = 1$, with and without using a length unit ρ_0 .

The use of Feynman's "jewel" **C** (or rather **H**) by geometry and physics has not found its full extent and rank due to these handicaps. For instance $Z \to Z'$ $= W.Z$ can be "metrically continued" both as a Lorentz transformation of special relativity and as a rotation of Euclidean geometry if complementary coordinate descriptions of **C** are accepted.
S. de Leo emphasized "The complexified quaternionic reformulation of special relativity is … a choice and not a necessity ", l996 [4]. With de Leo I sense, "that quaternions are the natural candidate to describe special relativity". With Anderson and Joshi I think that the usual ways to use quaternions in relativity (and quantum mechanics) is a non-straightforward one. My way to understand relativity with the help of Feynman's jewel ***C*** (and ***H***) is straightforward but diametrical: It is a prejudice that "the field of quaternions is a four-dimensional Euclidean space". We should not generalize "real" quaternions to "complex" ones to overcome this "Euclidean form". We have to realize that *a real unit* for measuring trigonometric angles and Cartesian coordinates is a choice and not a necessity. Mathematics supplies physics with both an imaginary and a real unit for measuring angles and the space-coordinates of physical events. Still the component form (6.3), (6.4) of the E-function – *even now free of coordinates* - possesses the *figure* of a Lorentz transformation. And the E-function *in linear and non-periodical coordinates is* the Lorentz transformation.

Schrödinger has proved that complex Ψ-numbers and his Ψ-function are physical realities. Still in the twenties he had difficulties to accept that his Ψ-numbers are *genuine complex numbers* ([15], p. 171. Cf. footnote 5 of [14]). Theoretical physics discussed the "Mathematical foundations of quantum mechanics". (Cf. for example the English translation of J. von Neumann's investigation [7]). But has science sufficiently discussed, why theoretical physics (beginning with quantum mechanics) is using *complex* numbers in a very sweeping extend? Does only nowaday for instance R. Penrose start to see the Ψ-numbers as genuine complex physical realities? [8, 9]

Many very imaginary units puff out of words and formulae of some quantum mechanical textbooks. In my nightmares they appear like cycles and epicycles in a non-Copernicus world. Science has to discuss new coordinate systems. Angles generate more fundamental frames than lengths.

Can science only build a stable bridge between relativity and quantum physics if the geometrical understanding of complex numbers is more generalized, and the complementary coordinate descriptions of these algebraic *and* geometric entities is practised in this way that these elements, perceived as physical quanta [14], sufficiently mirror the physical duality of waves and particles?

I thought it important to touch possible consequences of my view of numbers to the fundamentals of theoretical physics. But I want to emphasize that the content of this article is pure mathematics. Only a bad Bourbaki scholar may guess that geometrical pictures of numbers must not be a theme of pure mathematics. Essential steps of mathematical progress can be seen as fruits of quarrels between the theory of numbers and the concept of geometry. The form of Euclid's Elements is essentially conditioned by the difficulty that the diagonal of a unit-square can only be measured by $\sqrt{2}$. Modern real analysis was only possible after Descartes' idea of analytic geometry. The understanding of mathematics as a theory of structures is founded on the discovery of non-Euclidean geometries. So I want to clarify the concept and the position of Natural Geometry in answering two questions:

1. *Does natural geometry of numbers be a non-Euclidean geometry?*

YES:

- This geometry can start without using Pythagoras' theorem.
- In the beginning this geometry does not possess Euclidean straight lines and lengths on these lines.

NO:

- The sum of angles in conformal triangles is equal to two right angles, as in Euclidean triangles.
- Natural geometry generalizes the Euclidean concept of similarity (Ähnlichkeit) and shares this idea with Euclidean geometry and only with this geometry.

2. *Does natural geometry of numbers be a non-Cartesian geometry?*

YES:

- This geometry starts without using Cartesian coordinate systems.
- In the beginning a number is not separated in its real and imaginary parts. Coordinates are not generally used to describe the imaginary space-parts.

NO:

- Natural geometry uses numbers to describe geometrical figures. It uses this basic idea of Descartes.
- Natural geometry looks out for a perfection of Descartes' basic idea by replacing real numbers with complex ones, by 'unification' of the theory of numbers and the geometry of Euclidean triangles, in seeing the geometric world as **H-**number-line.

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